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Order Reduction for Models of Space Structures Using Modal Cost Analysis

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Modal cost analysis furnishes a promising methodology for developing dynamical models of space structures for use in control systems analysis. Economy and accuracy can be attained by only retaining vibration modes that contribute significantly to an appropriately defined "cost" function. Expressions for modal costs (especially simple for "lightly damped" structures) are derived for attitude control, vibration suppression, and shape control. These techniques are illustrated through application to a high-order finite element model of a large platform-type structure.

Introduction

IT is a basic premise of this paper that mathematical models for engineering systems often need to be reduced in size (i.e., in order) prior to the application of multivariable control theory, and that a rational method should be used for carrying out this order reduction. If the model is linear and the control objective is formulated in terms of a quadratic cost functional, one such method is modal cost analysis,[1] wherein the total cost is expressed as a sum of "modal costs," with each modal cost being associated with a suitable constituent "mode" of the system. In a paper[2] companion to the present one, modal cost analysis (MCA) is illustrated for linear matrix-second-order systems, a class to which many mechanical systems belong. The purpose of the present paper is to apply these results[2] to flexible spacecraft model reduction by deriving literal modal-cost expressions in terms of mode shapes, frequencies, and damping ratios. For illustration, three basic control objectives will be considered: attitude control; vibration suppression; and shape control.

Types of Modes

To clarify the use of the term "mode" in modal cost analysis, a brief discussion of modes is in order. Four types of modes are illustrated for flexible spacecraft in Fig. 1. The natural modes of vibration for constituent flexible appendages and the whole spacecraft are depicted in Figs. 1a and 1b, respectively. These modes are calculated by the methods of structural dynamics, and each such "vibration" mode in the model raises the system order by two. When control devices (sensors and actuators) are added (Fig. 1c), the "modes" associated with these devices must also be considered. However, these modes might be called "Jordan" modes, for each of which the system order is augmented by one. Prior to establishing feedback, the system model (Fig. 1c) includes the spacecraft vibration modes and the Jordan modes of the control devices. Finally, after feedback, the closed-loop Jordan modes can be defined (see Fig 1d).

Figure 1 also implies a hierarchy of mode types, with additional elements of the ultimate model being added at successive stages. Although intuition and the incomplete evidence in the literature suggest that order reduction should always be postponed to as late a stage as possible, it is recognized that a degree of order reduction may be necessary at each stage. Modal cost analysis[2] is applicable at each stage. This paper focuses on order reduction for vehicle modes (Fig. 1b).

Sources of Coupling

A recurring theme in modal cost analysis is coupling between coordinates. Three types of coupling can be identified: dynamical coupling, input coupling, and output coupling.

Dynamical coupling is the intrinsic coupling between the coordinates of the structural dynamical model. As an example, consider the simple mechanical system:

$$\ddot{q}_1 + 0.1\ddot{q}_2 + q_1 = u \qquad 0.1\ddot{q}_1 + \ddot{q}_2 + 10q_2 = 10u \tag{1}$$

Here, q_1 and q_2 are generalized coordinates (perhaps the appendage modal coordinates of Fig. 1a), and u might represent a control force. The system order is to be reduced from four to two. Based on the 10% inertial coupling between

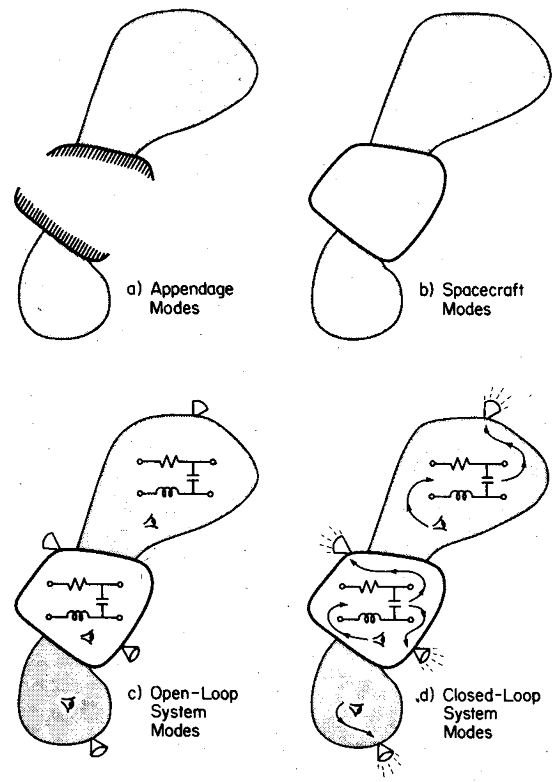

Fig. 1 Four types of modes in flexible spacecraft.

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q_1 and q_2, one might naively§ delete q_2 to form the reduced model

$$\ddot{q}_1 + q_1 = u \tag{2}$$

This truncation of an appendage mode may be inferior to first finding the dynamically uncoupled coordinates (vehicle modes, Fig. 1b) via the transformation to new coordinates, η_1 and η_2,

$$q_1 = 0.999\eta_1 - 0.112\eta_2 \qquad q_2 = 0.011\eta_1 + 1.005\eta_2 \tag{3}$$

whereupon the model transforms to

$$\begin{aligned} \ddot{\eta}_1 + (1-0.001)\eta_1 &= (1+0.110)u \\ \ddot{\eta}_2 + (10+0.112)\eta_2 &= (10-0.617)u \end{aligned} \tag{4}$$

to three significant digits. Truncation can now proceed in terms of η_1 and η_2.

In the foregoing, it has been tacitly assumed that the input u is not considered in the model reduction process. However, if $u = u(q_1, q_2, \dot{q}_1, \dot{q}_2)$ or $u = u(\eta_1, \eta_2, \dot{\eta}_1, \dot{\eta}_2)$, a new source of coupling is present—input coupling—and this coupling should be taken into account if the presupposed feedback law is known. Input coupling is not directly treated in this paper. However, the performance measure which the control eventually seeks to minimize is of concern in this paper. In this way, those modes that will be of importance in the specific control objectives are determined. This important link between the modeling problem and the control problem is designed to reduce the bad effects of control after model reduction.

The third type of coupling is output coupling. In the present context, this coupling occurs in the following way: a quadratic cost functional V of the outputs is constructed that reflects the objective of the control system. When this functional is expanded in terms of modal coordinates, there are cross-coupling terms generally present.[1,2] It has been shown,[2] however, that these output-coupled terms in V are arbitrarily small under the assumption of "lightly-damped" structures. This assumption will be made in this paper.

To summarize, this paper eliminates *dynamical* coupling though the use of "spacecraft modes" (Fig. 1b), and *output* coupling through the "light-damping" assumption. We assume to be dealing with a very large number of modes, so that only open-loop analysis is possible. For treatment of closed-loop effects see Refs. 2 and 8.

General Flexible Spacecraft Model

Attention is now drawn to the following matrix-second-order system, which represents a broad class of flexible spacecraft control problems:

$$\mathfrak{M}\ddot{q} + \mathfrak{D}\dot{q} + \mathfrak{K}q = \mathfrak{N}w \tag{5a}$$

$$y = \mathcal{P}q + \mathcal{P}'\dot{q} \tag{5b}$$

$$\mathcal{E}\{w(t)\} = 0 \qquad \mathcal{E}\{w(t)w^T(\tau)\} = W\delta(t-\tau) \tag{5c}$$

$$V = \lim_{t\to\infty} \mathcal{E}\{y^T Q y\} \tag{5d}$$

The system inertia, damping, and stiffness matrices are $(\mathfrak{M}, \mathfrak{D}, \mathfrak{K})$, and white noise disturbance $w(t)$ has intensity W. $\mathfrak{N}$ is a noise-distribution matrix to be specified in greater detail presently. The output $y(t)$ in Eq. (5b) is the output to be suppressed by the control system, in accordance with minimizing the cost functional V in Eq. (5d) with $Q > 0$. The dimensions of the vectors w and y are respectively n_w and n_y, and the dimensions of $\mathfrak{N}, \mathcal{P}, \mathcal{P}'$, W and Q are readily inferred. The model represented by Eq. (5) may be thought of as being associated with the appendage modes of Fig. 1a.

We further assume that $\mathfrak{M}^T = \mathfrak{M} > 0$, $\mathfrak{D}^T = \mathfrak{D} \geq 0$, $\mathfrak{K}^T = \mathfrak{K} \geq 0$. It follows that there exists a transformation $q = \mathfrak{T}\eta$ that simultaneously diagonalizes $\mathfrak{M}$ and $\mathfrak{K}$:

$$\mathfrak{T}^T\mathfrak{M}\mathfrak{T} = 1 \qquad \mathfrak{T}^T\mathfrak{K}\mathfrak{T} = \hat{\mathfrak{K}} = \text{diag}\{0, \ldots, 0, \omega_1^2, \ldots, \omega_N^2\} \tag{6}$$

Where **1** is the unit matrix, and N is the number of vehicle-mode natural frequencies. The zero frequencies in Eq. (6) are a consequence of the "rigid" modes for the spacecraft. In Eq. (6) and throughout, the rigid modes are not numbered. Applying the transformation T to the system of Eq. (5), we obtain the model expressed in vehicle modal coordinates:

$$\ddot{\eta}_r = \mathfrak{T}_r^T \mathfrak{N} w \tag{7a}$$

$$\ddot{\eta}_e + \mathfrak{D}_e\dot{\eta}_e + \omega^2\eta_e = \mathfrak{T}_e^T\mathfrak{N}w \tag{7b}$$

$$y = \mathcal{P}\mathfrak{T}_r\eta_r + \mathcal{P}\mathfrak{T}_e\eta_e + \mathcal{P}'\mathfrak{T}_r\dot{\eta}_r + \mathcal{P}'\mathfrak{T}_e\dot{\eta}_e \tag{7c}$$

where $\omega = \text{diag}\{\omega_1, \ldots, \omega_N\}$ and the following partitionings have been employed

$$\eta = [\eta_r^T \eta_e^T]^T \qquad \mathfrak{T} = [\mathfrak{T}_r, \mathfrak{T}_e] \qquad \mathfrak{T}^T\mathfrak{D}\mathfrak{T} = \begin{bmatrix} 0 & 0 \\ 0 & \mathfrak{D}_e \end{bmatrix} \tag{8}$$

The subscripts r and e denote vehicle rigid modes and vehicle elastic modes, respectively. $\mathfrak{D}_e$ is a symmetric damping matrix representing dissipation in the flexible structure. Material damping or friction at the joints of a truss are two examples.

This paper is restricted to *open*-loop considerations, specifically, to inspecting the interface between structural modeling and the control objectives. Any undamped "rigid modes" that are observable[3] in the vector y in Eq. (5b) will result in an infinite modal cost for these modes [see Eq. (12) below]. Retaining them in the reduced model is therefore consistent with the rule of retaining modes with the largest modal costs. Since only finite modal costs need be calculated, only the elastic modes, η_e, need be explicitly treated. Thus the system under consideration is

$$\ddot{\eta}_e + \mathfrak{D}_e\dot{\eta}_e + \omega^2\eta_e = \mathfrak{N}_e w \tag{9a}$$

$$y = \mathcal{P}_e\eta_e + \mathcal{P}_e'\dot{\eta}_e \tag{9b}$$

$$V = \lim_{t\to\infty} \mathcal{E}\{y^T Q y\} \tag{9c}$$

where

$$\mathfrak{N}_e \triangleq \mathfrak{T}_e^T\mathfrak{N} \qquad \mathcal{P}_e \triangleq \mathcal{P}\mathfrak{T}_e \qquad \mathcal{P}_e' \triangleq \mathcal{P}'\mathfrak{T}_e \tag{9d}$$

The stage is now set for a modal cost analysis of this system of equations, which represents a broad class of three-axis controlled flexible spacecraft and a variety of control objectives.

Modal Cost Analysis for Lightly Damped Flexible Spacecraft

When model cost analysis is applied to general linear-invariant systems, it is found[1] that the cost V can be expressed as a sum of "modal costs," where the modal cost associated with mode α includes some contribution from the other modes. Under the assumption of "lightly damped" modes, however, where the open-loop poles lie very near to the imaginary axis in the s-plane, it has been shown[2] that the

§It might *not* be naive to truncate appendage modal coordinates. In fact, this is *standard* procedure in complex structures composed of several substructures. The "assembly" of the model occurs after truncation of each substructure model.

modal cost associated with mode α depends solely on mode α asymptotically as the damping approaches zero. The "lightly damped" assumption is appropriate for space structures.

To make this statement precise, let

$$\mathfrak{D}_e = \mathrm{diag}\{2\zeta_1\omega_1,...,2\zeta_N\omega_N\} \tag{10}$$

in Eq. (9a), with $\zeta_\alpha \ll 1$, for $\alpha = 1,...,N$. It has been shown elsewhere[2] that

$$V = \sum_{\alpha=1}^{N} V_\alpha \tag{11}$$

where, asymptotically as $\zeta_\alpha \to 0$, an analytical solution to the covariance equation for Eq. (9) yields

$$V_\alpha = (p_\alpha^T Q p_\alpha + \omega_\alpha^2 p_\alpha'^T Q p_\alpha')\sigma_\alpha^2 / 4\zeta_\alpha\omega_\alpha^3 \qquad (\alpha = 1,...,N) \tag{12}$$

where

$$[p_1 \cdots p_N] \triangleq \mathcal{P}_e \qquad [p_1' \cdots p_N'] \triangleq \mathcal{P}_e' \tag{13}$$

$$\sigma_\alpha^2 \triangleq \{\mathfrak{N}_e W \mathfrak{N}_e^T\}_{\alpha\alpha} \tag{14}$$

Special cases of Eq. (12) have an even simpler form if only one of $\{\mathcal{P}_e, \mathcal{P}_e'\}$ is present in Eq. (9b), i.e., if only displacement or rate is to be suppressed by the controller.

To apply this formulation to more specific classes of problems, it is necessary to specify the nature of the disturbance inputs (in terms of $\mathfrak{N}_e$ and W), and the control objectives (in terms of $\mathcal{P}_e, \mathcal{P}_e'$, and Q). It is difficult to be general about these specifications because they tend to be mission dependent. However, some progress can be made under reasonable assumptions.

To begin, we rewrite the scalar form of the modal equations (9a) as

$$\ddot{\eta}_\alpha + 2\zeta_\alpha\omega_\alpha\dot{\eta}_\alpha + \omega_\alpha^2\eta_\alpha = \gamma_\alpha(t) \qquad (\alpha = 1,...,N) \tag{15}$$

In other words, $\gamma(t) \triangleq \mathfrak{N}_e w$ and $\gamma^T = [\gamma_1 \ldots \gamma_N]$. It is known from the theory of structural dynamics that

$$\gamma_\alpha(t) = \int_E \phi_\alpha^T(r) f(r,t)\,\mathrm{d}r \qquad (\alpha = 1,...,N) \tag{16a}$$

where $f(r,t)$ is the force per unit volume at r, at time t, dr is an element of volume at r, and the domain of integration, E, consists of all the flexible portions of the spacecraft. The vehicle mode shapes ϕ_α satisfy the following orthonormality conditions:

$$\int_V \phi_\alpha^T \phi_\beta \,\mathrm{d}m = \delta_{\alpha\beta} \tag{16b}$$

Note that the integration in Eq. (16b), unlike in (16a), is over the mass of the vehicle. In view of Eqs. (16), the disturbance distribution matrix $\mathfrak{N}_e$ in Eq. (9) is determined once $f(r,t)$ is known. Two possibilities for f are now considered: a distributed environmental force, and noise from control actuators.

Distributed Disturbances

The temporal dependence of $f(r,t)$ is assumed to be white noise

$$f(r,t) = F(r)w(t) \qquad \mathcal{E}[w(t)w^T(\tau)] = W\delta(t-\tau) \tag{17}$$

The simplest spatial distribution, $F(r)$, is a uniform one, and the next simplest is one linear in r. Indeed, these two distributions appear as the first two terms in a Taylor series expansion of $F(r)$. Thus, the ith column of the $3 \times n_w$ matrix $F(r)$ has the following Taylor series expansion:

$$F_i(r) = F_i(0) + \mathcal{J}_i(0)r + \ldots \tag{18}$$

where $\mathcal{J}_i(0)$ is the Jacobian of $F_i(r)$ evaluated at $r = 0$:

$$\mathcal{J}_i(0) = \left.\frac{\partial F_i(r)}{\partial r}\right|_{r=0} = \frac{\partial F_i(0)}{\partial r} \tag{19}$$

Now from Eqs. (14-17)

$$\sigma_\alpha^2 = f_\alpha^T W f_\alpha = \|f_\alpha\|_W^2 \tag{20}$$

where

$$f_\alpha^T \triangleq \int_E \phi_\alpha^T(r)F(r)\,\mathrm{d}r \tag{21}$$

Substituting Eq. (18) in Eq. (21), the ith element of the n_w-dimensional vector f_α is

$$(f_\alpha)_i = \int_E \phi_\alpha^T(r)\,[F_i(0) + \mathcal{J}_i(0)r]\,\mathrm{d}r \tag{22}$$

Since $\|f_\alpha\|$ decreases with α (although not always monotonically),[4] and W is independent of the mode number, σ_α^2 in Eq. (20) tends to decrease with α.

Noise from Control Actuators

A further example of disturbances is the noise introduced unintentionally via the control actuators. For this class of disturbances, the "noise-distribution matrix" $\mathfrak{N}$ in Eq. (5a) is just the "control-distribution matrix," often denoted by the symbol $\mathcal{B}$. For n_u force actuators located at positions $r_1,...,r_{n_u}$ throughout the domain $\mathcal{E}$, we have ($\eta_w = \eta_u$ in this case):

$$f(r,t) = \sum_{i=1}^{n_u} a_i w_i(t)\delta(r - r_i) \tag{23}$$

where a_i and w_i are, respectively, a unit vector in the direction of the ith control force, and the noise from that control force. From Eqs. (16a) and (23)

$$\gamma_\alpha(t) = \sum_{i=1}^{n_u} \phi_\alpha^T(r_i)a_i w_i(t) \tag{24}$$

Consequently,

$$(f_\alpha)_i = \phi_\alpha^T(r_i)a_i \tag{25}$$

Substituting Eq. (25) in Eq. (20) the variance σ_α^2 for mode α is obtained.

Another possibility is that the control commands are exercised through torque actuators (CMG's, for example). With this setup, it is known that[3]

$$\gamma_\alpha(t) = \sum_{i=1}^{n_u} \psi_\alpha^T(r_i)a_i w_i(t) \tag{26}$$

where $\psi_\alpha(r_i)$ represents the three (small) angles associated with the vehicle mode shape ϕ_α at r_i, and a_i is a unit vector defining the axis of the ith control torque. Comparing Eq. (26) with Eq. (24), it is seen that the results for force actuators in the last paragraph can be interpreted also for torque actuators by substituting ψ_α for ϕ_α.

We return now to Eq. (12), the pivotal result used in this paper. It is already understood that ζ_α and ω_α are the damping ratio and natural frequency associated with the αth mode of the space structure, and the last four paragraphs have discussed possible sources of the disturbance variances, σ_α^2. Further sources can no doubt be modeled for specific missions. To complete a discussion of the modal cost V_α for the lightly damped mode α, attention is now turned to the control objectives, as embodied in the matrices $\mathcal{P}_e$, $\mathcal{P}_e'$, Q.

Attitude Control

Suppose the aim of the control system is to reduce to zero the attitude error θ of some part of the space structure. In general,

$$\theta = \theta_r + \theta_f \tag{27}$$

where $\theta_r(t)$ is the attitude history with no structural deformations, $\eta_e \equiv 0$ and θ_f is the additional attitude error due to flexibility:

$$\theta_f = \sum_{\alpha=1}^{N} \theta_\alpha \eta_\alpha(t) \tag{28}$$

where θ_α reflects the participation of mode α in θ. In attitude control objectives y will be defined so that at least some of the rigid body modes will be observable in y. Hence, these modes are selected for retention in the reduced model to be used for controller design, and modal cost analysis is not needed to make that decision. In fact, the modal cost is infinite for rigid modes, which are observable in y and disturbable from u, w [put $\omega_\alpha = 0$ in Eq. (12)]. Thus, modal cost analysis is needed only to reduce the number of structural modes in the model. With attitude control in mind, one defines, in the spirit of Eq. (9b),

$$y = [\theta_f^T \quad \dot{\theta}_f^T]^T \tag{29}$$

$$Q = \mathrm{diag}\{c_\theta Q_\theta, c_\theta' Q_\theta\} \qquad (c_\theta, c_\theta' \geq 0) \tag{30}$$

admitting the possibility that attitude rate might also be important. For attitude rate control only, choose $c_\theta = 0$, $c_\theta' = 1$. The 3×3 matrix Q_θ quantizes the importance of the three angular elements in θ. The cost functional implied by Eqs. (9c), (29), and (30) is

$$V = \lim_{t\to\infty} \mathcal{E}\{c_\theta \theta_f^T Q_\theta \theta_f + c_\theta' \dot{\theta}_f^T Q_\theta \dot{\theta}_f\} \tag{31}$$

It follows from Eq. (12) that the modal cost for mode α is

$$V_\alpha = (c_\theta + c_\theta' \omega_\alpha^2)(\theta_\alpha^T Q_\theta \theta_\alpha / 4\zeta_\alpha \omega_\alpha^3)\sigma_\alpha^2 \tag{32}$$

For example, if all three attitude angles are equally important in a three-axis attitude control task (so that Q_θ is the unit matrix), and if attitude rates are unimportant,

$$V_\alpha = \|\theta_\alpha\|^2 \sigma_\alpha^2 / 4\zeta_\alpha \omega_\alpha^3 \tag{33}$$

In any case, the relative importance of the vehicle modes is established by rearranging them so that $V_1 \geq V_2 \geq \ldots \geq V_N$. The "dominant mode" in modal cost analysis is the mode with the largest modal cost. Note from Eq. (32) that the modal costs drop off faster with ω_α if attitude rates are unimportant ($c_\theta' = 0$).

Vibration Suppression

A second control task of considerable interest is vibration suppression. To express this control objective in terms of modal coordinates, we note that the contributions of mode α to the kinetic and potential energies are $\dot{\eta}_\alpha^2$ and $\omega_\alpha^2 \eta_\alpha^2$, respectively. Thus, a sensible choice (but not the only one) for a cost measure is:

$$y^T Q y = \sum_{\alpha=1}^{N} (c_\alpha \omega_\alpha^2 \eta_\alpha^2 + c_\alpha' \dot{\eta}_\alpha^2) \qquad (c_\alpha, c_\alpha' \geq 0) \tag{34}$$

where c_α and c_α' are modal weighting constants. Comparing Eq. (34) with Eqs. (9b), (9c), and (13), we find:

$$V_\alpha = (c_\alpha + c_\alpha')\sigma_\alpha^2 / 4\zeta_\alpha \omega_\alpha \tag{35}$$

Again, the contribution of mode α to V has been expressed in terms of known (or chosen) modal parameters. Interestingly, the relative weighting assigned to the potential and kinetic energy contributions from mode α is immaterial; only the sum $c_\alpha + c_\alpha'$ enters the modal cost V_α. Furthermore, *if* one happens to weight all modes the same, c_α and c_α' will be independent of α, so that

$$V_\alpha = \sigma_\alpha^2 / \zeta_\alpha \omega_\alpha \tag{36}$$

where one recalls that $-\zeta_\alpha \omega_\alpha$ is the real part of the characteristic root corresponding to mode α.

Figure (Shape) Control

Another family of control problems of current interest is generated by the demands of radio telescopes and phased-array antennas. For these applications, one is interested in the deformations throughout the flexible structure. The relative importance of each vehicle mode can be stipulated by the cost functional:

$$y^T Q y = \sum_{\alpha=1}^{N} (c_\alpha \eta_\alpha^2 + c_\alpha' \dot{\eta}_\alpha^2) \qquad (c_\alpha, c_\alpha' \geq 0) \tag{37}$$

In this criterion, c_α and c_α' measure the contribution of mode α to the deterioration of the control objective. They might, for example, be geometrically derived factors for a flexible phased-array antenna, or a line-of-sight error in a flexible telescope. Under current assumptions the cost contribution for mode α is

$$V_\alpha = (c_\alpha + c_\alpha' \omega_\alpha^2)\sigma_\alpha^2 / 4\zeta_\alpha \omega_\alpha^3 \tag{38}$$

A specific example of c_α and c_α' for a large platform type structure will be given in the example to follow. The important point to note is that modal cost analysis is a useful tool in ranking the relative importance of vehicle vibration modes, particularly for lightly damped structures.

Some comments on the role of damping ratio, ζ_α, in reduced order structural modeling is required. The modal cost Eq. (12), (which measures the importance of the mode α in quadratic performance measures), contains the modal damping ratio ζ_α, which is a poorly known parameter in all space structures. Our knowledge (or lack thereof) of the modal damping does not alter the fact that the importance of a mode depends upon its damping. It is therefore a fact that one's ability to construct "good" reduced models is not separable from one's ability to assess the relative damping from mode to mode. Conversely, Eq. (12) also illustrates the fact that modal damping does *not* influence modal importance (modal "rankings" by modal cost), if all modes have the same damping ratio $\xi_\alpha = \bar{\zeta}$ for all α.

Summary of Modal Cost Analysis

The modal cost expression in Eq. (12), and its specialized versions in Eqs. (32), (35), and (38), provide a basis for open-loop model order reduction for lightly damped structures. Taking Eqs. (9) and (10) as the starting point, the following algorithm summarizes the required steps.

Step I: Specify the structure. That is, specify the modal data $\{\omega_\alpha, \zeta_\alpha, \alpha = 1, \ldots, N\}$ and $\mathfrak{J}_e$.

Step II: Specify the excitation expected. That is, specify $\{\mathfrak{N}_e, W\}$, and calculate σ_α^2 from Eq. (14). Or, specify σ_α^2 directly.

Step III: Specify the control objectives. That is, specify $\{\mathcal{P}_e, \mathcal{P}_e', Q\}$.

Step IV: Calculate the modal costs V_α from Eq. (12). Or, for attitude control, vibration suppression, or figure control, calculate V_α from Eqs. (32), (35), or (38), respectively.
Step V: Arrange the modes in descending order according to their contribution to the cost functional V. That is, arrange them so that $V_1 \geq V_2 \geq \ldots \geq V_N$. Then keep the first r modes, where $r<N$ is the maximum number that can be accommodated in the control system design or synthesis. The other $N-r$ modal coordinates are discarded, by dropping the associated modal equations in Eq. (9a).

Example: A Generic Space Structure Model

Figure 2 shows a 12.5 km × 5 km space structure, for which a distributed-parameter model has been generated.[4] At the center of the elastic truss structure E, which has mass m_e and moments of inertia I_{ex}, I_{ey}, is located a rigid body $\mathcal{R}$ of mass m_r and moments of inertia I_{rx}, I_{ry}. The mass of $\mathcal{R}$ is 10% of the mass of E (i.e., $m_r/m_e=0.1$), and $I_{rx}/I_{ex}=I_{ry}/I_{ey}=0.01$. The structure equivalent to E (mass/area ρ) is essentially two-dimensional, and its first 97 elastic modes have been computed.[4] A brief description of its motion is as follows. The general motion of the vehicle is a superposition of the vehicle modes thus:

$$\begin{bmatrix} z(t) \\ \theta_x(t) \\ \theta_y(t) \\ u(x,y,t) \end{bmatrix} = \begin{bmatrix} Z(t) \\ \Theta_x(t) \\ \Theta_y(t) \\ 0 \end{bmatrix} + \sum_{\alpha=1}^{97} \begin{bmatrix} z_\alpha \\ \theta_{x\alpha} \\ \theta_{y\alpha} \\ u_\alpha(x,y) \end{bmatrix} \eta_\alpha(t) \tag{39}$$

where $z(t)$, $\theta_x(t)$, $\theta_y(t)$ are, respectively, the displacement in the z-direction and the rotations about the x- and the y-axis of the rigid body $\mathcal{R}$. $u(x,y,t)$ is the elastic deflection of any arbitrary point $(x,y)\in E$. Comparing Eq. (39) with Eq. (7a) the three rigid modal coordinates $Z(t)$, $\Theta_x(t)$, $\Theta_y(t)$ correspond to the $\eta_r(t)$ and $\eta_e^T \triangleq [\eta_1 \ldots \eta_N]$. The total deflection $w(x,y,t)$ of any point p in the vehicle is

$$w(x,y,t) = z + y\theta_x - x\theta_y + \begin{cases} 0 & p\in\mathcal{R} \\ u(x,y,t) & p\in E \end{cases} \tag{40}$$

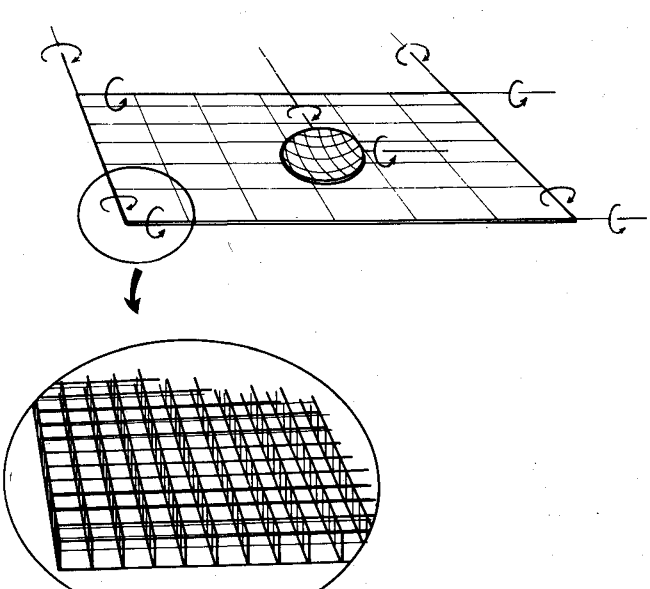

Fig. 2 Model of a generic space structure.

Similarly, the vehicle mode ϕ_α is related to $u_\alpha(x,y)$ in Eq. (39) as follows:

$$\phi_\alpha(x,y,t) = z_\alpha + y\theta_{x\alpha} - x\theta_{y\alpha} + \begin{cases} 0 & p\in\mathcal{R} \\ u_\alpha(x,y,t) & p\in E \end{cases} \tag{41}$$

where ϕ_α obeys the orthonormality conditions, Eq. (16b). With the aid of Eqs. (39) and (41) w has the following expansion

$$w(x,y,t) = Z(t) + y\Theta_x(t) - x\Theta_y(t) + \sum_{\alpha=1}^{97} \phi_\alpha(x,y)\eta_\alpha(t) \tag{42}$$

Each mode is assumed to have the same but unknown damping ratio.

Modal cost analysis is now illustrated for the four control problems posed, combining two control objectives and two disturbance models. The labels I and II refer to the control objective, and the labels A and B to the disturbance model.

Two Control Tasks

Attitude Control (I). In this problem the control objective is to regulate θ, the attitude of $\mathcal{R}$. Equations (27-32) are applicable. Here, $y=[\theta_{xf}\theta_{yf}\dot\theta_{xf}\dot\theta_{yf}]^T$, where $\theta_{xf}=\Sigma\theta_{x\alpha}\eta_\alpha$ and $\theta_{yf}=\Sigma\theta_{y\alpha}\eta_\alpha$. We choose, in Eq. (30), $c_\theta=1$, $c_\theta'=\beta$, $Q_\theta=\text{diag}\{I_{xx},I_{yy}\}$. Therefore, for this attitude control example, the modal costs are:

$$V_\alpha(\text{I}) = (1+\beta\omega_\alpha^2)(I_{xx}\theta_{x\alpha}^2 + I_{yy}\theta_{y\alpha}^2)\sigma_\alpha^2/(4\zeta_\alpha\omega_\alpha^3) \tag{43}$$

Shape Control (II). In this problem, it is desired to regulate the shape of the truss structure to remain flat. To reflect this objective the following cost functional may be chosen

$$V(\text{II}) \triangleq \lim_{t\to\infty} \mathcal{E}\int_V [w_f^2(x,y,t) + \beta\dot w_f^2(x,y,t)]\,dm$$
$$w_f \triangleq \Sigma\phi_\alpha(x,y)\eta_\alpha(t) \tag{44}$$

V(II) concerns the flatness of the elastic structure because integration is carried over the entire vehicle. Utilizing Eqs. (39-42), and the orthogonality condition Eq. (16b) obeyed by ϕ_α and u_α, the V(II) in Eq. (44) becomes

$$V(\text{II}) = \lim_{t\to\infty} \mathcal{E}(\eta_e^T\eta_e + \beta\dot\eta_e^T\dot\eta_e) \tag{45}$$

Comparing Eq. (37) with Eq. (45), one gets $c_\alpha=1$, $c_\alpha'=\beta$, $(\alpha=1,\ldots,N)$. Therefore the modal costs for this problem are

$$V_\alpha(\text{II}) = (1+\beta\omega_\alpha^2)\sigma_\alpha^2/(4\zeta_\alpha\omega_\alpha^3) \tag{46}$$

Disturbance Models

Two types of disturbances are now considered for each of the above control tasks.

Distributed Disturbances (A). For illustration $F(r)$ in Eq. (17) can be chosen to be

$$F(r) = (1, y, -x) \qquad r^T = (x, y) \tag{47}$$

Substituting Eq. (47) in Eq. (21), one gets

$$f_\alpha^T = \frac{1}{\rho}\left(\frac{m_r}{m}p_\alpha, \quad \frac{I_{rx}}{I_{xx}}h_{x\alpha}, \quad \frac{I_{ry}}{I_{yy}}h_{y\alpha}\right) \tag{48}$$

where the modal momentum coefficients[5] P_α, $h_{x\alpha}$, $h_{y\alpha}$ are defined by

$$\frac{1}{\rho}\left(\frac{m_r}{m}p_\alpha, \quad \frac{I_{rx}}{I_{xx}}h_{x\alpha}, \quad \frac{I_{ry}}{I_{yy}}h_{y\alpha}\right) \triangleq \int_E \phi_\alpha^T(1,y,-x)\,dA \tag{49}$$

A 3×1 noise vector $w(t)$ compatible with Eq. (47) has intensity $W=\text{diag}\{w_{11}^2,\ w_{22}^2,\ w_{33}^2\}$. Consequently, with the aid of Eq. (20),

$$\sigma_\alpha^2(\text{A})=\left\{\left(\frac{m_r}{m}\right)^2 w_{11}^2 p_\alpha^2+\left(\frac{I_{rx}}{I_{xx}}\right)^2 w_{22}^2 h_{x\alpha}^2+\left(\frac{I_{ry}}{I_{yy}}\right)^2 w_{33}^2 h_{y\alpha}^2\right\} \qquad (\alpha=1,\ldots,N) \qquad (50)$$

For computations we assume that the variances in Eq. (50) are such that

$$\left(\frac{m_r}{m}\right)^2 w_{11}^2=\left(\frac{I_{rx}}{I_{xx}}\right)^2 w_{22}^2=\left(\frac{I_{ry}}{I_{yy}}\right)^2 w_{33}^2=\text{const} \qquad (51)$$

The *actual* values of w_{11}, w_{22}, w_{33} need not be known in this case.

Noisy Actuators (B). For illustration purposes, let a pair of torquers about the x- and y-axes be mounted on each of the four corners of the structure and on the rigid body in Fig. 2. Assuming uncorrelated white noise due to the noisy actuators the intensity can be written as

$$W=\text{diag}\{w_{x1}^2,w_{y1}^2,w_{x2}^2,\ldots,w_{y5}^2\}=50\,(\text{Nm})^2\,1 \qquad (52)$$

where w_{xi}^2, $w_{yi}^2\,(i=1,\ldots,5)$ are the intensities of the disturbance torques generated by the actuators $u_1,u_2,\ldots,u_{10}$ in Fig. 2. With the aid of Eqs. (20) and (26) it is clear that

$$\sigma_\alpha^2(\text{B})=\sum_{i=1}^{5}\left[\psi_{x\alpha}^2(x_i,y_i)\,w_{yi}^2+\psi_{y\alpha}^2(x_i,y_i)\,w_{xi}^2\right] \qquad (53)$$

where

$$\psi_{x\alpha}\triangleq\frac{\partial\phi_\alpha}{\partial y}=\theta_{x\alpha}+\frac{\partial u_\alpha}{\partial y} \qquad \psi_{y\alpha}\triangleq\frac{\partial\phi_\alpha}{\partial x}=-\theta_{y\alpha}+\frac{\partial u_\alpha}{\partial x} \qquad (54)$$

The deriatives of ϕ_α denoted by ψ_α in Eq. (26) represent the slopes to be evaluated at the locations $(x_i,\ y_i,\ i=1,\ldots,5)$ of the actuators. Corresponding numerical results are discussed in the next section.

Results and Discussion

The maximum modal cost in Eqs. (43) and (46) is normalized to unity and modal costs below 10^{-6} are set to 10^{-6} in Figs. 3 and 4. The frequencies plotted are nondimensional; the dimensional frequencies are obtained by multiplying frequencies in the abscissa by $(D/\rho a^4)^{1/2}$ where the flexural rigidity $D=20\times 10^8$ N, the mass density $\rho=0.2623$ kg/m^2, and the length of the structure is $a=12{,}500$ m. Figure 3 deals with attitude control in the presence of either distributed disturbances or noisy actuators. It is observed that although the modal cost reduces with frequency, the trend is not monotonic and the cost is distributed over many more modes

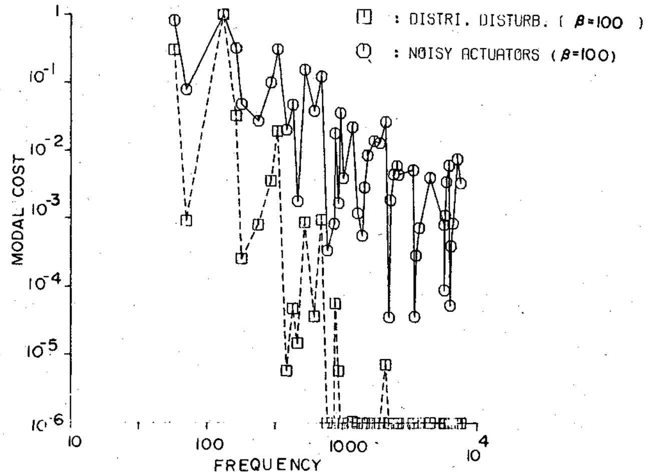

Fig. 3 Modal cost vs frequency for attitude control.

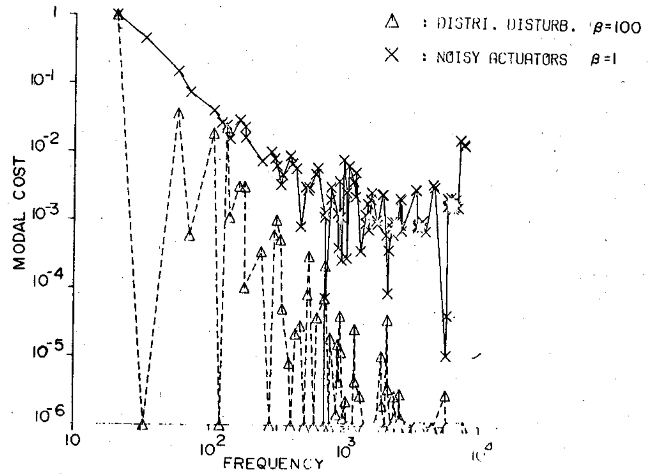

Fig. 4 Modal cost vs frequency for shape and attitude control.

Table 1 Generic space structure model: mode sequence and predicted model error index

			Mode sequence (r = modes retained)																	
			←Largest modal cost															Smallest modal cost→		
Control objective	Disturbance	Weighting scalar β	$r=$ 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16...	Predicted model error index	
Attitude (I)	Distributed (A)	$0\leq\beta\leq 1$	7	3	9	16	14	4	12	11	23	27	20	26	33	21	18	36...	0.00001	
		100	7	3	9	16	14	27	4	23	12	11	33	20	26	21	57	36...	0.00001	
	Noisy actuators (B)	$0\leq\beta\leq 1$	3	7	9	16	4	14	11	23	12	27	20	18	26	37	33	43...	0.00069	$(\beta=0)$
		100	7	3	9	16	23	27	14	4	11	20	26	37	12	57	43	18...	0.00137	$(\beta=1)$
																			0.03584	$(\beta=100)$
Attitude and shape (II)	Distributed (A)	$0\leq\beta\leq 1$	1	3	7	5	9	10	8	4	15	14	12	16	11	24	29	17...	0.00003	$(\beta=0)$
		100	1	3	7	5	9	10	8	15	14	4	16	12	24	29	11	23...	0.00005	$(\beta=1)$
																			0.0004	$(\beta=100)$
	Noisy actuators (B)	0	1	2	3	4	5	9	6	7	10	11	8	13	18	14	12	19...	0.04745	$(\beta=0)$
		1	1	2	3	4	5	9	6	7	10	11	94	95	8	96	97	13...	0.08964	$(\beta=1)$
		100	94	95	96	97	1	2	76	77	67	68	85	81	92	83	37	86...	0.36361	$(\beta=100)$

in the case of noisy actuators than in the case of distributed disturbances. Note from Figs. 3 and 4 that the critical modes are not ordered by frequency. This is due to the fact that the modal cost depends upon the relative degree of observability and disturbability of the mode in addition to its damping and frequency. The importance of attitude rate in the control objectives is dictated by β, which is 100 in Fig. 3. Table 1 shows that for attitude control, the mode sequence is not very sensitive to the choice of β.

In Fig. 4, both shape and attitude control are desired; and the *control objective* is Eq. (45). For distributed disturbances the selection of critical modes is again not sensitive to β (see also Table 1). However, for noisy actuators this is not the case. Table 1 shows that if β is chosen high enough ($\beta = 100$ in this case) the higher frequency modes become critical. This might be expected since a large β reflects a dominating concern for high rates, and the higher rates attend the higher frequencies. Of course, this last result in Table 1 is difficult to apply in practice because the highest modes are the least well known. However, this result is shown in order to make the point that such limitations are the result of modeling errors (parameter errors or colored noise disturbances) not treated in this paper, not of the MCA technique itself. Any of the following three means can remove this difficulty:

1) Use more realistic damping models. There is evidence[6] that the higher modes have a larger damping ratio ζ_α. In fact, the damping for the higher modes is suggested to be related to higher powers of modal frequency $(\omega_i, \omega_i^2, \omega_i^3)$. If such models $\zeta_\alpha = (k_1\omega_\alpha + k_2\omega_\alpha^2 + k_3\omega_\alpha^3)$ were used in the modal cost formula, Eq. (12), the denominator would change from $\{4\zeta_\alpha\omega_\zeta^3\}$ to $\{4\omega_\alpha^3(k_1\omega_\alpha + k_2\omega_\alpha^2 + k_3\omega_\alpha^3)\}$. In this event, all high-frequency modes in Figs. 3 and 4 would have substantially smaller modal costs.

2) Utilize frequency weighted disturbance models. (Let σ_α^2 in Eqs. (12) and (14) drop off with frequency to reflect the realistic and finite bandwidths of the disturbances.)

3) Do not use large rate penalties (β) in the performance criterion, Eq. (45).

Figure 5 shows how many modes, r, must be retained in the reduced model in order to keep the "predicted model error index" less than any specified number, where

$$\text{Predicted Model Error Index} \triangleq \sum_{\alpha=r+1}^{97} \frac{V_\alpha}{V} \qquad V = \sum_{\alpha=1}^{97} V_\alpha \quad (55)$$

It is observed that the error index reduces more rapidly for the distributed disturbances than for the noisy actuators. For example, to acquire a model error index of 0.01, four elastic modes are needed for the attitude control and five for the shape control, both in the presence of distributed disturbances; whereas 30 and 70 elastic modes, respectively, are needed in the case of noisy actuators. The sequences of the retained modes are of course different in all four cases. It is evident that disturbances propagating through the actuators have the potential to excite more modes of the structure, distributing the performance cost over many modes. Also, for a given type of disturbance and a specified model error index, shape control requires more modes in the design model than attitude control would require.

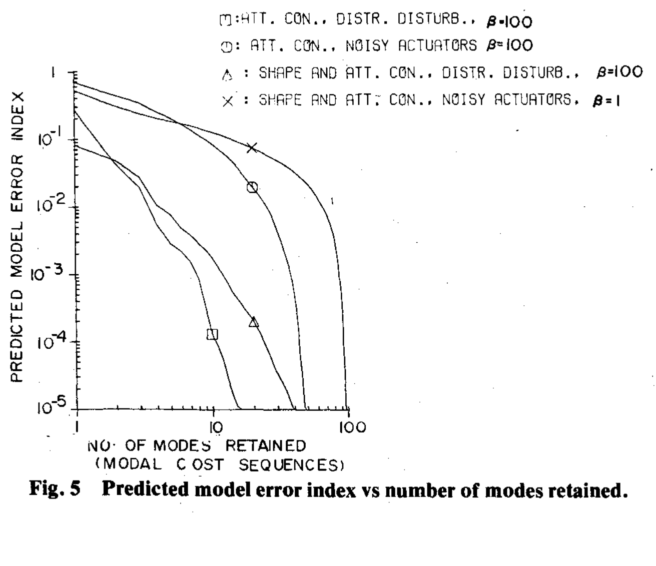

Fig. 5 Predicted model error index vs number of modes retained.

Concluding Remarks

Modal cost analysis (MCA) provides a means by which control objectives and disturbance characteristics can be integrated into model reduction decisions. This is accomplished by decomposing a suitable quadratic cost functional into individual modal contributions called "modal costs." This decomposition is particularly straightforward for "lightly damped" space structures. Literal expressions for the modal costs have been derived for three classes of control tasks—attitude control, vibration suppression, and shape control—although the method is applicable to any control objective. Two sources of noise excitation have also been discussed—externally imposed disturbances, and noise from the control actuators—and reasonably general formulas are derived for the modal disturbance intensities. It is clear from these results that the natural frequencies are not the only modal parameters of importance. The modal damping, the control-related weightings, and the disturbance intensities are equally important. It is also evident that different groups of modes should be selected for different control tasks.

These general results have been illustrated for a representative two-dimensional space structure. Calculations further underline the fact that the *number* of modes needed and the *best* modes to choose depend critically upon a precise statement of the control objective and upon the character of the disturbances. For example, due to their potential for exciting many modes, noisy actuators may necessitate more control software than would more benign environmental disturbances. Finally, it is demonstrated that, other things being equal, shape control requires higher order models than does attitude control.

Modal cost analysis can also be used for closed-loop control system design, but this is beyond the scope of this paper.

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